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Equations of Fluctuating Nonlinear Hydrodynamics for Normal Fluids

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The full set of fluctuating nonlinear hydrodynamic equations for normal fluids is derived from the conventional Langevin equations extended to include multiplicative noise. The equations describing the set of conserved variables (the mass density ρ, the momentum density $\mathbf{g}$, the energy density ε) agree with those found by Morozov for a case of a driving free energy which is a local function of the hydrodynamic variables. We show here that if the standard form of the hydrodynamic equations is to hold in the absence of noise, then the driving free energy must be a local function of $\mathbf{g}$ and ε, but it may have to be a nonlocal function of the mass density.

KEY WORDS: Langevin equations; multiplicative noise; thermodynamics.

1. INTRODUCTION

Fluctuating nonlinear hydrodynamics has long been established as the accepted approach for treating nonlinear effects in a wide variety of physical systems. The so-called "mode-coupling" nonlinearities have been important in treating the behavior of simple fluids(1,2) and liquid crystals.(3) There are important nonlinear effects near a critical point(4) and also in broken symmetry regimes.(5) The appropriate equations are believed(6) to be asymptotically exact in the large-distance, long-time limit. It is expected,(7) for example, that the Navier–Stokes equation, including noise, gives a full description of macroscopic turbulence in an incompressible fluid.

Although the full nonlinear equations of motion for a simple fluid in the absence of fluctuating forces are well known,(8) the corresponding problem in the presence of noise has attracted little attention in the

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literature.[2] Landau and Lifshitz first suggested a form for the full set of fluctuating hydrodynamic equations for a fluid (ref. 10; also see the last chapter of ref. 8). While it was clear that the equations with Gaussian noise proposed by Landau and Lifshitz were correct in the linear regime, they left unanswered the question of the self-consistency of these equations in the nonlinear regime. Van Saarloos *et al.*(11) later showed that the assumption of simple Gaussian noise is *not* consistent with thermodynamics and the standard constitutive relations. Thus, for example, in the Landau and Lifshitz scheme, one cannot simultaneously satisfy thermodynamics and Fourier's law. Zubarev and Morozov(12) and Morozov(13) finally showed how the inconsistencies described above can be eliminated by generalizing the description to include multiplicative noise.

In this paper we discuss how the development due to Morozov can be understood from the point of view of generalized Langevin equations. As part of our analysis here we point out that Morozov's analysis is restricted to the case of the driving free energy that is a local function of the fluctuating variables. This means that one cannot, without further development, include the standard square-gradient terms in the free energy. Our main new result is that only nonlocal and/or gradient terms involving the mass density can be incorporated into the driving free energy for a simple fluid without changing the basic structure of the equations. The free energy must be a local functional of the momentum and energy densities if the conventional structure is to be preserved.

2. GENERALIZED LANGEVIN EQUATIONS

In this section we present the generalized Langevin equation description[3] including the possibility of multiplicative noise. The set of fields $\psi_\alpha(\mathbf{x}, t)$ defined in d spatial dimensions are assumed to obey the Langevin equation

$$\frac{\partial\psi_\alpha(\mathbf{x}, t)}{\partial t} = V_\alpha[\psi] - \sum_\beta \int d^d y\, \Gamma^{\alpha\beta}[\mathbf{x}, \mathbf{y}; \psi]\, \frac{\delta F[\psi]}{\delta\psi_\beta(\mathbf{y})} + \sum_{\beta\mu} \int d^d y\, G^{\alpha\beta\mu}[\mathbf{x}, \mathbf{y}; \psi]\, \Theta_{\beta\mu}(\mathbf{y}, t) \tag{2.1}$$

where $V_\alpha[\psi]$ is the streaming velocity

$$V_\alpha[\psi] = \{\psi_\alpha(\mathbf{x}), F[\psi]\} \tag{2.2}$$

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[2] The fluctuating nonlinear hydrodynamic equations suppressing the energy density was first obtained by Enz and Turski.(9)

[3] For a standard treatment of the generalized Langevin equations, see Ma and Mazenko.(14)

where $\{A, B\}$ is the Poisson bracket of A and B and $F[\psi]$ is the associated effective free energy. The matrices Γ and G in (2.1) are related to the dissipative processes in the system and $\Theta_{\beta\mu}(\mathbf{x}, t)$ is the Gaussian noise, which is assumed to obey the second moment

$$\langle \Theta_{\alpha\beta}(\mathbf{x}, t)\, \Theta_{\mu\nu}(\mathbf{y}, t') \rangle = 2T_0 D_{\alpha\beta,\mu\nu}\,\delta(\mathbf{x}-\mathbf{y})\,\delta(t-t') \tag{2.3}$$

where T_0 is the ambient temperature and $D_{\alpha\beta,\mu\nu}$ is a symmetric matrix independent of ψ, $\mathbf{x}$, and t and dependent on the bare transport coefficients in the system. If the matrix G in (2.1) is dependent on ψ, then one has multiplicative noise.

We now seek the conditions under which the Langevin equation (2.1) is compatible with the equilibrium probability distribution

$$P[\psi] \sim e^{-F[\psi]/T_0} \tag{2.4}$$

This answer follows from the use of the Fokker–Planck description for this problem.

Consider the quantity

$$g_\phi(t) = \prod_\alpha \prod_{\mathbf{x}} \delta(\phi_\alpha(\mathbf{x}) - \psi_\alpha(\mathbf{x}, t)) \tag{2.5}$$

and its average

$$P_\phi(t) = \langle g_\phi(t) \rangle \tag{2.6}$$

We want to construct (2.1) such that in the long-time equilibrium limit

$$\lim_{t \to \infty} P_\phi(t) = P_{EQ}[\phi] = N e^{-F[\phi]/T_0} \tag{2.7}$$

The time dependence of $P_\phi(t)$ is governed by the generalized Fokker–Planck operator. We can construct this quantity by first taking the time derivative of $P_\phi(t)$ and using the chain rule for differentiation

$$\frac{\partial P_\phi(t)}{\partial t} = -\sum_\alpha \int d^d x \left\langle \frac{\delta g_\phi(t)}{\delta \phi_\alpha(\mathbf{x})} \frac{\partial \psi_\alpha(\mathbf{x}, t)}{\partial t} \right\rangle \tag{2.8}$$

Using (2.1) and the δ-function structure of $g_\phi(t)$, we obtain

$$\begin{aligned} \frac{\partial P_\phi(t)}{\partial t} = &-\sum_\alpha \int d^d x \frac{\delta}{\delta \phi_\alpha(\mathbf{x})} \left\{ \left[V_\alpha[\mathbf{x}, \phi] - \sum_\beta \int d^d y\, \Gamma^{\alpha\beta}[\mathbf{x}, \mathbf{y}; \phi] \frac{\delta F}{\delta \phi_\beta(\mathbf{y})} \right] P_\phi(t) \right. \\ &\left. + \sum_{\beta\mu} \int d^d y\, G^{\alpha\beta\mu}[\mathbf{x}, \mathbf{y}; \phi] \langle \Theta_{\beta\mu}(\mathbf{y}, t)\, g_\phi(t) \rangle \right\} \end{aligned} \tag{2.9}$$

Evaluation of the correlation of the Gaussian noise with $g_\phi(t)$ follows from the use of the identity

$$\langle \Theta_{\beta\mu}(\mathbf{y}, t)\, g_\phi(t)\rangle = 2T_0 \sum_{\sigma\gamma} D_{\beta\mu,\sigma\gamma} \left\langle \frac{\delta g_\phi(t)}{\delta \Theta_{\sigma\gamma}(\mathbf{y}, t)} \right\rangle \tag{2.10}$$

Again using the chain rule, we find

$$\langle \theta_{\beta\mu}(\mathbf{y}, t)\, g_\phi(t)\rangle = -2T_0 \sum_{\sigma,\gamma,\nu} D_{\beta\mu,\sigma\gamma} \int d^dz \frac{\delta}{\delta\phi_\nu(\mathbf{z})} \left\langle g_\phi(t) \frac{\delta\psi_\nu(\mathbf{z}, t)}{\delta\Theta_{\sigma\gamma}(\mathbf{z}, t)} \right\rangle \tag{2.11}$$

As discussed by Ma and Mazenko,$^{(14)}$ the derivative of the field with respect to the noise at equal times follows from the equation of motion (2.1) as

$$\frac{\delta\psi_\nu(\mathbf{z}, t)}{\delta\Theta_{\sigma\gamma}(\mathbf{y}, t)} = \frac{1}{2} G^{\nu\sigma\gamma}[\mathbf{z}, \mathbf{y}; \psi] \tag{2.12}$$

and

$$\langle \Theta_{\beta\mu}(\mathbf{y}, t)\, g_\phi(t)\rangle = -T_0 \sum_{\sigma\gamma\nu} D_{\beta\mu,\sigma\gamma} \int d^dz \frac{\delta}{\delta\phi_\nu(\mathbf{z})} \{G^{\nu\sigma\gamma}[\mathbf{z}, \mathbf{y}; \phi]\, P_\phi(t)\} \tag{2.13}$$

The equation of motion for $P_\phi(t)$ can then be written in the form

$$\frac{\partial P_\phi(t)}{\partial t} = D_\phi P_\phi(t) \tag{2.14}$$

where the Fokker–Planck operator is given by

$$\begin{aligned} D_\phi = -\sum_\alpha \int d^dx \frac{\delta}{\delta\phi_\alpha(\mathbf{x})} \Bigg\{ & V_\alpha[\mathbf{x}, \phi] - \sum_\beta \int d^dy\, \Gamma^{\alpha\beta}[\mathbf{x}, \mathbf{y}; \phi] \frac{\delta F}{\delta\phi_\beta(\mathbf{y})} \\ & - T_0 \sum_{\beta\mu,\sigma\gamma} \int d^dy\, d^dz\, G^{\alpha\beta\mu}[\mathbf{x}, \mathbf{y}; \phi]\, D_{\beta\mu,\sigma\gamma} \frac{\delta}{\delta\phi_\nu(\mathbf{z})} G^{\nu\sigma\gamma}[\mathbf{z}, \mathbf{y}; \phi] \Bigg\} \end{aligned} \tag{2.15}$$

The condition for equilibrium is given by

$$D_\phi e^{-F[\phi]/T_0} = 0 \tag{2.16}$$

or

$$\begin{aligned} -\sum_\alpha \int d^dx \frac{\delta}{\delta\phi_\alpha(\mathbf{x})} \Bigg\{ & V_\alpha[\mathbf{x}, \phi] - \sum_\beta \int d^dy\, \Gamma^{\alpha\beta}[\mathbf{x}, \mathbf{y}; \phi] \frac{\delta F}{\delta\phi_\beta(\mathbf{y})} \\ & - T_0 \sum_{\beta\mu,\sigma,\gamma} \int d^dy\, d^dz\, G^{\alpha\beta\mu}[\mathbf{x}, \mathbf{y}; \phi]\, D_{\beta\mu,\sigma\gamma} \\ & \times \frac{\delta}{\delta\phi_\nu(z)} G^{\nu\sigma\gamma}[\mathbf{z}, \mathbf{y}; \phi] \Bigg\} e^{-F[\phi]/T_0} = 0 \end{aligned} \tag{2.17}$$

We restrict our analysis here to the case where this reduces to three independent conditions:

(i) The divergence condition

$$\sum_{\alpha} \int d^d x \frac{\delta}{\delta \phi_\alpha(\mathbf{x})} \left(V_\alpha[\mathbf{x}, \phi]\, e^{-F[\phi]/T_0} \right) = 0 \tag{2.18}$$

(ii) The locality condition

$$\sum_{\nu} \int d^d z \frac{\delta}{\delta \phi_\nu(\mathbf{z})} G^{\nu\sigma\gamma}[\mathbf{z}, \mathbf{x}; \phi] = 0 \tag{2.19}$$

(iii) The detailed balance condition

$$\Gamma^{\alpha\beta}[\mathbf{x}, \mathbf{y}; \phi] = \sum_{\sigma\gamma, \mu\nu} \int d^d z\, G^{\alpha\sigma\gamma}[\mathbf{x}, \mathbf{z}; \phi]\, D_{\sigma\gamma, \mu\nu} G^{\beta\mu\nu}[\mathbf{y}, \mathbf{z}; \phi] \tag{2.20}$$

We can make a few general comments here. The divergence condition can be rewritten in the form

$$\begin{aligned} &\sum_{\alpha} \int d^d x \frac{\delta}{\delta \phi_\alpha(\mathbf{x})} \left(\{\phi_\alpha(\mathbf{x}), F[\phi]\}\, e^{-F[\phi]/T_0} \right) \\ &\quad = e^{-F[\phi]/T_0} \sum_{\alpha} \int d^d x \frac{\delta}{\delta \phi_\alpha(\mathbf{x})} \{\phi_\alpha(\mathbf{x}), F[\phi]\} \\ &\quad\quad - \sum_{\alpha} \int d^d x\, \{\phi_\alpha(\mathbf{x}), F[\phi]\} \frac{\delta F[\phi]}{\delta \phi_\alpha(\mathbf{x})} \frac{1}{T_0} e^{-F[\phi]/T_0} \end{aligned} \tag{2.21}$$

But

$$\begin{aligned} &\sum_{\alpha} \int d^d x\, \{\phi_\alpha(\mathbf{x}), F[\phi]\} \frac{\delta F[\phi]}{\delta \phi_\alpha(\mathbf{x})} \\ &\quad = \sum_{\alpha, \beta} \int d^d x\, d^d y\, \{\phi_\alpha(\mathbf{x}), \phi_\beta(\mathbf{y})\} \frac{\delta F[\phi]}{\delta \phi_\alpha(\mathbf{x})} \frac{\delta F[\phi]}{\delta \phi_\beta(\mathbf{y})} \\ &\quad = 0 \end{aligned} \tag{2.22}$$

due to the antisymmetry of the Poisson brackets. The divergence condition reduces then to

$$\sum_{\alpha} \int d^d x \frac{\delta}{\delta \phi_\alpha(\mathbf{x})} V_\alpha[\mathbf{x}, \phi] = 0 \tag{2.23}$$

In general this is not zero and depends on the nature of the Poisson brackets.

For most applications one can assume that the matrix elements G are independent of ϕ and the locality condition is trivially satisfied. When one includes the energy variable this is no longer possible, as we shall see, and condition (ii) becomes more nontrivial.

The detailed balance condition shows that Γ is symmetric since, from (2.3), D is symmetric,

$$\Gamma^{\alpha\beta}[\mathbf{x}, \mathbf{y}; \phi] = \Gamma^{\beta\alpha}[\mathbf{y}, \mathbf{x}; \phi] \tag{2.24}$$

3. SINGLE-VARIABLE CASE

We can gain some appreciation for the general case by considering the simple case of a single conserved field, the energy density $\varepsilon(\mathbf{x}, t)$, which is driven by a free energy

$$F = E - T_0 S = \int d^d x\, [\varepsilon(\mathbf{x}) - T_0 s(\mathbf{x})] \tag{3.1}$$

where E is the total energy, S is the entropy, and $s(\mathbf{x})$ is the entropy density.

The local temperature is defined by

$$\frac{1}{T(\mathbf{x})} = \frac{\delta S}{\delta\varepsilon(\mathbf{x})} \tag{3.2}$$

The constitutive relation associated with the conserved energy is Fourier's law

$$\mathbf{J}_\varepsilon = -\lambda_0 \nabla_{\mathbf{x}} T(\mathbf{x}) \tag{3.3}$$

where $\mathbf{J}_\varepsilon$ is the energy current in the absence of noise and λ_0 is the "bare" thermal conductivity.

In this case of a single scalar field the streaming velocity is zero and the Langevin equation (2.1) takes the form

$$\frac{\partial\varepsilon(\mathbf{x}, t)}{\partial t} = -\int d^d y\, \Gamma[\mathbf{x}, \mathbf{y}; \varepsilon] \frac{\delta F}{\delta\varepsilon(\mathbf{y})} + \sum_i \int d^d y\, G^i[\mathbf{x}, \mathbf{y}; \varepsilon]\, \theta_i(\mathbf{y}, t) \tag{3.4}$$

and the detailed balance condition becomes

$$\Gamma[\mathbf{x}, \mathbf{y}; \varepsilon] = \sum_{ij} \int d^d z\, G^i[\mathbf{x}, \mathbf{z}; \varepsilon]\, D^0_{ij} G^j[\mathbf{y}, \mathbf{z}; \varepsilon] \tag{3.5}$$

Since the energy is conserved, the simplest choice for the quantity G is

$$G^i[\mathbf{x}, \mathbf{z}; \varepsilon] = \nabla^i_{\mathbf{x}}[f(\varepsilon)\,\delta(\mathbf{x}-\mathbf{z})] \tag{3.6}$$

and

$$\Gamma[\mathbf{x}, \mathbf{y}; \varepsilon] = \sum_{ij} \nabla^i_{\mathbf{x}} \nabla^j_{\mathbf{y}} \left[f^2(\varepsilon(\mathbf{x}))\, D^0_{ij} \delta(\mathbf{x}-\mathbf{y}) \frac{\delta F}{\delta\varepsilon(\mathbf{y})} \right] \tag{3.7}$$

The energy current in the absence of noise is given then by

$$\begin{aligned} J^i_\varepsilon(\mathbf{x}) &= \sum_j \int d^d y\, \nabla^j_{\mathbf{y}} [f^2(\varepsilon(\mathbf{x}))\, D^0_{ij} \delta(\mathbf{x}-\mathbf{y})] \frac{\delta F}{\delta\varepsilon(\mathbf{y})} \\ &= -\sum_j f^2(\varepsilon(\mathbf{x}))\, D^0_{ij} \nabla^j_{\mathbf{x}} \left(1 - \frac{T_0}{T(\mathbf{x})}\right) \\ &= -T_0 \frac{f^2(\varepsilon(\mathbf{x}))}{T^2(\mathbf{x})} \sum_j D^0_{ij} \nabla^j_{\mathbf{x}} T(\mathbf{x}) \end{aligned} \tag{3.8}$$

We have a reconciliation with Fourier's law (3.3) if

$$D^0_{ij} = \frac{\lambda_0}{T_0} \delta_{ij} \tag{3.9}$$

and

$$f(\varepsilon(\mathbf{x})) = T(\mathbf{x}) \tag{3.10}$$

Thus thermodynamics and Fourier's law demand that the noise be multiplicative,

$$G^i[\mathbf{x}, \mathbf{y}; \varepsilon] = \nabla^i_{\mathbf{x}}[T(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})] \tag{3.11}$$

and the Langevin equation takes the form

$$\frac{\partial\varepsilon(\mathbf{x}, t)}{\partial t} = \lambda_0 \nabla^2_{\mathbf{x}} T(\mathbf{x}, t) + \sum_i \nabla^i_x [T(\mathbf{x}, t)\, \theta_i(x, t)] \tag{3.12}$$

The locality condition (2.19) then takes the form

$$\begin{aligned} \int d^d x \frac{\delta}{\delta\varepsilon(\mathbf{x})} G^i[\mathbf{x}, \mathbf{y}; \varepsilon] &= \int d^d x \frac{\delta}{\delta\varepsilon(\mathbf{x})} \nabla^i_{\mathbf{x}}[T(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})] \\ &= \int d^d x \frac{\delta T(\mathbf{y})}{\delta\varepsilon(\mathbf{x})} \nabla^i_{\mathbf{x}} \delta(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{3.13}$$

We have called (2.19) the locality condition because, if $S(\varepsilon)$ is a local function of ε (no gradient terms), then T is also a local function of ε and

$$\frac{\delta T(\mathbf{y})}{\delta\varepsilon(\mathbf{x})}=\frac{\partial T(\mathbf{y})}{\partial\varepsilon(\mathbf{y})}\,\delta(\mathbf{x}-\mathbf{y}) \tag{3.14}$$

and

$$\int d^dx\,\frac{\delta}{\delta\varepsilon(\mathbf{x})}\,G^i[\mathbf{x},\mathbf{y};\varepsilon]=\frac{\partial T(\mathbf{y})}{\partial\varepsilon(\mathbf{y})}\int d^dx\,\delta(\mathbf{x}-\mathbf{y})\,\nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y})=0 \tag{3.15}$$

The importance of this last result has been emphasized by Morozov.[4] If S is a function of the gradients of ε, then (2.19) is not generally satisfied. In that case one can guarantee equilibration only by adding an additional term to the right-hand side of (2.1) which just cancels the term associated with (2.19) in (2.17). Such terms change the basic structure of the constitutive relation.

We might expect the multiplicative nonlinearity in the Langevin equation (3.12) to contribute to the renormalization of the thermal conductivity. But in turns out that this is not the case. We can easily show using the Fokker–Planck development$^{(14)}$ that the Fourier–Laplace transform of the energy correlation function has the usual pole structure in the low-frequency, long-wavelength limit:

$$C(\mathbf{q},z)=\frac{\rho_0 c_v}{z+iD_T q^2} \tag{3.16}$$

where ρ_0 is the ambient mass density, c_v the specific heat, and the diffusivity D_T can be written as

$$D_T=\frac{\lambda_R}{\rho_0 c_v}=\lim_{q\to 0}\frac{\lambda_0\langle\delta T(\mathbf{q})\,\delta\varepsilon(-\mathbf{q})\rangle}{\langle\delta\varepsilon(\mathbf{q})\,\delta\varepsilon(-\mathbf{q})\rangle} \tag{3.17}$$

with λ_R being the renormalized thermal conductivity. (3.17) becomes trivial if we use the identity in the canonical ensemble

$$\frac{\partial\langle A\rangle}{\partial T_0}=\frac{1}{T_0^2}\langle\delta A\,\delta E\rangle \tag{3.18}$$

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[4] While the meaning of the integral in (3.15) is ambiguous in the continuum description, Morozov shows that in a rigorous treatment of the theory, defined on a lattice, the integral in (3.15) vanishes.

By setting, in turn, $A = T(\mathbf{x})$ and $\varepsilon(\mathbf{x})$, one obtains

$$\begin{aligned} &\lim_{q\to 0} \langle \delta T(\mathbf{q})\, \delta\varepsilon(-\mathbf{q})\rangle = \langle \delta T(\mathbf{x})\, \delta E\rangle = T_0^2 \\ &\lim_{q\to 0} \langle \delta\varepsilon(\mathbf{q})\, \delta\varepsilon(-\mathbf{q})\rangle = \langle \delta\varepsilon\, \delta E\rangle = T_0^2 \frac{\partial}{\partial T_0}\left(\frac{E}{V}\right) = T_0^2 \rho_0 c_v \end{aligned} \tag{3.19}$$

and (3.17) reduces to

$$\lambda_R = \lambda_0 \tag{3.20}$$

4. NORMAL FLUID

4.1. Equilibrium Structure

The first step in applying the development in Section 2 to a fluid is to specify the set of fields $\psi_\alpha(\mathbf{x})$ and the associated free energy $F[\psi]$. From a dynamic point of view the appropriate set of independent variables are the conserved densities $\psi_\alpha(\mathbf{x}) = (\rho(\mathbf{x}), \mathbf{g}(\mathbf{x}), \varepsilon(\mathbf{x}))$, where ρ is the mass density, $\mathbf{g}$ the momentum density, and ε the energy density. In the case of a microcanonical ensemble the associated free energy is $F[\psi] = -S[\psi]$, where S is the entropy. If the entire system is in contact with a heat bath with a uniform temperature T_0, then the appropriate free energy is the Helmholtz free energy

$$F[\psi] = E - T_0 S[\psi] \tag{4.1}$$

where

$$E = \int d^d x\, \varepsilon(\mathbf{x}) \tag{4.2}$$

is the total energy. We shall assume throughout that we are working in the canonical ensemble where (4.1) gives the free energy driving the Langevin equations.

From a thermodynamic point of view it is usually preferable to work in the representation where the independent variables are given by the set $\phi_\alpha(\mathbf{x}) = (\rho(\mathbf{x}), \mathbf{g}(\mathbf{x}), s(\mathbf{x}))$, where $s(\mathbf{x})$ is the entropy density and the associated potential is now the energy E. In particular, if E is a function of ϕ_α, we can define the local intensive variables

$$G_\alpha(\mathbf{x}) = \frac{\delta E}{\delta\phi_\alpha(\mathbf{x})} \tag{4.3}$$

Thus,

$$G_s(\mathbf{x}) = \frac{\delta E}{\delta s(\mathbf{x})} \equiv T(\mathbf{x}) \tag{4.4}$$

where $T(\mathbf{x})$ is the local temperature,

$$G_{g_i}(\mathbf{x}) = \frac{\delta E}{\delta g_i(\mathbf{x})} = V_i(\mathbf{x}) \tag{4.5}$$

where $V_i(\mathbf{x})$ is the local velocity, and

$$G_\rho(\mathbf{x}) = \frac{\delta E}{\delta \rho(\mathbf{x})} = \mu(\mathbf{x}) \tag{4.6}$$

where $\mu(\mathbf{x})$ is the local chemical potential. The local pressure is then defined by the local version of the Euler relation

$$\begin{aligned} P(\mathbf{x}) &= \sum_\alpha \phi_\alpha(\mathbf{x})\, G_\alpha(\mathbf{x}) - \varepsilon(\mathbf{x}) \\ &= \sum_\alpha \phi_\alpha(\mathbf{x}) \frac{\delta E}{\delta \phi_\alpha(\mathbf{x})} - \varepsilon(\mathbf{x}) \\ &= \rho(\mathbf{x})\, \mu(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \mathbf{V}(\mathbf{x}) + s(\mathbf{x})\, T(\mathbf{x}) - \varepsilon(\mathbf{x}) \end{aligned} \tag{4.7}$$

There is a key point to be made at this stage. If E is a nonlocal function of the ϕ_α, i.e., contains gradient terms, then the entropy and energy representations will give different definitions of the intensive variables. In the entropy representation the intensive variables are defined by

$$\tilde{G}_\alpha(\mathbf{x}) = \frac{\delta S}{\delta \psi_\alpha(\mathbf{x})} \tag{4.8}$$

In a purely thermodynamic context

$$\tilde{G}_\varepsilon(\mathbf{x}) = \frac{1}{T(\mathbf{x})} = \frac{1}{G_s(\mathbf{x})} \tag{4.9}$$

and

$$\tilde{G}_\alpha(\mathbf{x}) = -\frac{1}{T(\mathbf{x})} G_\alpha(x) \tag{4.10}$$

for $\alpha \neq \varepsilon$ or s. When gradient terms are included, (4.9) and (4.10) are violated by terms proportional to the gradients. We will return to this point below.

4.2. Poisson Brackets

A key ingredient in specifying the Langevin equation (2.1) is the set of Poisson brackets among the variables $\psi_\alpha(\mathbf{x})$. In the case of a normal fluid we start with the set $\phi_\alpha(\mathbf{x}) = \{\rho(\mathbf{x}), \mathbf{g}(\mathbf{x}), s(\mathbf{x})\}$ and use the results for this set to construct the Poisson brackets for the set of variables $\psi_\alpha(\mathbf{x}) = \{\rho(\mathbf{x}), \mathbf{g}(\mathbf{x}), \varepsilon(\mathbf{x})\}$.

It is straightforward to show, starting from the microscopic definitions of the density and the momentum density, that ρ and $\mathbf{g}$ satisfy the set of Poisson brackets

$$\{\rho(\mathbf{x}), g_i(\mathbf{y})\} = -\nabla^i_{\mathbf{x}}[\delta(\mathbf{x}-\mathbf{y})\,\rho(\mathbf{x})] \tag{4.11}$$

$$\{g_i(\mathbf{x}), \rho(\mathbf{y})\} = \nabla^i_{\mathbf{y}}[\delta(\mathbf{x}-\mathbf{y})\,\rho(\mathbf{x})] \tag{4.12}$$

$$\{g_i(\mathbf{x}), g_j(\mathbf{y})\} = -\nabla^j_{\mathbf{x}}[\delta(\mathbf{x}-\mathbf{y})\,g_i(\mathbf{x})] + \nabla^i_{\mathbf{y}}[\delta(\mathbf{x}-\mathbf{y})\,g_j(\mathbf{x})] \tag{4.13}$$

$$\{\rho(\mathbf{x}), \rho(\mathbf{y})\} = 0 \tag{4.14}$$

It is assumed[5] that the entropy density transforms as a scalar under spatial translations and therefore has the same Poisson bracket structure as the mass density:

$$\{g_i(\mathbf{x}), s(\mathbf{y})\} = \nabla^i_{\mathbf{y}}[\delta(\mathbf{x}-\mathbf{y})\,s(\mathbf{x})] \tag{4.15}$$

$$\{\rho(\mathbf{x}), s(\mathbf{y})\} = 0 \tag{4.16}$$

$$\{s(\mathbf{x}), s(\mathbf{y})\} = 0 \tag{4.17}$$

The Poisson bracket relations for the set of variables ϕ_α can be written in the compact form

$$\{\phi_\alpha(\mathbf{x}), \phi_\beta(\mathbf{y})\} = \delta_{\alpha, g_i}\nabla^i_{\mathbf{y}}[\delta(\mathbf{x}-\mathbf{y})\,\phi_\beta(\mathbf{x})] - \delta_{\beta, g_i}\nabla^i_{\mathbf{x}}[\delta(\mathbf{x}-\mathbf{y})\,\phi_\alpha(\mathbf{x})] \tag{4.18}$$

Given the Poisson brackets for the ϕ_α, one can construct the Poisson brackets for the ψ_α using

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = \sum_{\mu, \nu} \int d^dx'\, d^dy'\, \frac{\delta\psi_\alpha(\mathbf{x})}{\delta\phi_\mu(\mathbf{x})} \frac{\delta\psi_\beta(\mathbf{y})}{\delta\phi_\nu(\mathbf{y})} \{\phi_\mu(\mathbf{x}), \phi_\nu(\mathbf{y})\} \tag{4.19}$$

Inserting (4.18) into (4.19) and doing the integration over the δ-function gives

$$\begin{aligned}\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = \sum_{i, \nu} \int d^dx' \Bigg[&\frac{\delta\psi_\beta(\mathbf{y})}{\delta\phi_\nu(\mathbf{x})} \nabla_{\mathbf{x}} \left\{ \frac{\delta\psi_\alpha(\mathbf{x})}{\delta g_i(\mathbf{x})} \phi_\nu(\mathbf{x}) \right\} \\ &- \frac{\delta\psi_\alpha(\mathbf{x})}{\delta\phi_\nu(\mathbf{x})} \nabla^i_{\mathbf{x}} \left\{ \frac{\delta\psi_\beta(\mathbf{y})}{\delta g_i(\mathbf{x})} \phi_\nu(\mathbf{x}) \right\} \Bigg]\end{aligned} \tag{4.20}$$

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[5] A derivation of Poisson brackets for the set $\phi(\mathbf{x}) = \{\rho, \mathbf{g}, s\}$ from the point of view of the symmetry is given by Dzyaloshinskii and Volovick.(15)

For ψ in the subset $\{\rho, \mathbf{g}\}$ these clearly reduce to their previous form. Thus we are left with the Poisson brackets for the energy variable. Unless we make some statements about the locality of the energy, we can make little progress. If we appeal to the principle of Galilean invariance, then we can conclude[6] that

$$E = \int d^d x \, \frac{g^2(\mathbf{x})}{2\rho(\mathbf{x})} + \int d^d x \, u(\mathbf{x}) \tag{4.21}$$

where u is a functional only of ρ and s, and

$$\frac{\delta\varepsilon(\mathbf{x})}{\delta g_i(\mathbf{y})} = \frac{g_i(\mathbf{x})}{\rho(\mathbf{x})} \, \delta(\mathbf{x}-\mathbf{y}) \equiv V_i(\mathbf{x}) \, \delta(\mathbf{x}-\mathbf{y}) \tag{4.22}$$

We have then

$$\{\rho(\mathbf{x}), \varepsilon(\mathbf{y})\} = -\sum_i \nabla^i_{\mathbf{x}} [\delta(\mathbf{x}-\mathbf{y}) \, g_i(\mathbf{x})] \tag{4.23}$$

$$\{g_i(\mathbf{x}), \varepsilon(\mathbf{y})\} = -\sum_\nu \phi_\nu(\mathbf{x}) \nabla^i_{\mathbf{x}} \frac{\delta\varepsilon(\mathbf{y})}{\delta\phi_\nu(\mathbf{x})} - \sum_j \nabla^j_{\mathbf{x}} [V_j(\mathbf{x}) \, g_i(\mathbf{x}) \, \delta(\mathbf{x}-\mathbf{y})] \tag{4.24}$$

and

$$\{\varepsilon(\mathbf{x}), \varepsilon(\mathbf{y})\} = -\sum_{i,\nu} V_i(\mathbf{x}) \, \phi_\nu(\mathbf{x}) \nabla^i_{\mathbf{x}} \frac{\delta\varepsilon(\mathbf{y})}{\delta\phi_\nu(\mathbf{x})} + \sum_{i,\nu} V_i(\mathbf{y}) \, \phi_\nu(\mathbf{y}) \nabla^i_{\mathbf{y}} \frac{\delta\varepsilon(\mathbf{x})}{\delta\phi_\nu(\mathbf{y})} \tag{4.25}$$

Further assumptions on the structure of $\varepsilon(\mathbf{x})$ are required to go further. Let us assume that ε is a local functional of $\phi_\alpha(\mathbf{x})$ and $\nabla^i_{\mathbf{x}} \phi_\alpha(\mathbf{x})$ only. Then

$$\frac{\delta\varepsilon(\mathbf{x})}{\delta\phi_\beta(\mathbf{y})} = \frac{\partial\varepsilon(\mathbf{x})}{\partial\phi_\beta(\mathbf{x})} \, \delta(\mathbf{x}-\mathbf{y}) + \sum_j \frac{\partial\varepsilon(\mathbf{x})}{\partial(\nabla^j_x \phi_\beta(\mathbf{x}))} \nabla^j_{\mathbf{x}} \delta(\mathbf{x}-\mathbf{y}) \tag{4.26}$$

Then, for example, if the pressure is given by

$$P(\mathbf{x}) = \sum_\alpha \phi_\alpha(\mathbf{x}) \, \frac{\delta E}{\delta\phi_\alpha(\mathbf{x})} - \varepsilon(\mathbf{x}) \tag{4.27}$$

one obtains

$$\nabla^i_{\mathbf{x}} P(\mathbf{x}) = \sum_\alpha \phi_\alpha(\mathbf{x}) \nabla^i_{\mathbf{x}} \frac{\delta E}{\delta\phi_\alpha(\mathbf{x})} - \sum_j \nabla^j_{\mathbf{x}} \left(\sum_\alpha \frac{\partial\varepsilon(\mathbf{x})}{\partial(\nabla^j_x \phi_\alpha(\mathbf{x}))} \nabla^i_{\mathbf{x}} \phi_\alpha(\mathbf{x}) \right) \tag{4.28}$$

---

[6] A semimicroscopic derivation of the kinetic part of (4.21) is given by Langer and Turski.(16)

Using this result, one can evaluate

$$\begin{aligned}&\sum_{\nu} \phi_\nu(\mathbf{x}) \nabla^i_{\mathbf{x}} \frac{\delta \varepsilon(\mathbf{y})}{\delta \phi_\nu(\mathbf{x})}\\ &\qquad = \nabla^i_{\mathbf{x}}[P(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})] + \varepsilon(\mathbf{x})\,\nabla^i_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) \sum_j \nabla^j_{\mathbf{x}} \Delta_{ij}(\mathbf{x})\end{aligned} \tag{4.29}$$

where

$$\Delta_{ij}(\mathbf{x}) = \sum_{\alpha} \frac{\partial \varepsilon(\mathbf{x})}{\partial(\nabla_j \phi_\alpha(\mathbf{x}))} \nabla^i_{\mathbf{x}} \phi_\alpha(\mathbf{x}) \tag{4.30}$$

This result, in turn, gives the final expressions for the Poisson brackets

$$\begin{aligned}\{g_i(\mathbf{x}), \varepsilon(\mathbf{y})\} = {}& -\nabla^i_{\mathbf{x}}[P(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})] - \sum_j \nabla^j_{\mathbf{x}}[V_j(\mathbf{x})\, g_i(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})]\\ &+ \nabla^i_{\mathbf{y}}[\varepsilon(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})] - \delta(\mathbf{x}-\mathbf{y}) \sum_j \nabla^j_x \Delta_{ij}(\mathbf{x})\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\{\varepsilon(\mathbf{x}), \varepsilon(\mathbf{y})\} = {}& -\sum_i \nabla^i_{\mathbf{x}}\{V_i(\mathbf{x})[\varepsilon(\mathbf{x}) + P(\mathbf{x})]\,\delta(\mathbf{x}-\mathbf{y})\}\\ &+ \sum_i \nabla^i_{\mathbf{y}}\{V_i(\mathbf{y})[\varepsilon(\mathbf{y}) + P(\mathbf{y})]\,\delta(\mathbf{x}-\mathbf{y})\}\end{aligned} \tag{4.32}$$

4.3. Streaming Velocities

The streaming velocities $V_\alpha[\psi]$ which go into the Langevin equation are given by

$$\begin{aligned}V_\alpha[\psi] &= \{\psi_\alpha(\mathbf{x}), F\}\\ &= \{\psi_\alpha(\mathbf{x}), E\} - T_0\{\psi_\alpha(\mathbf{x}), S\}\end{aligned} \tag{4.33}$$

We first note that

$$\{\psi_\alpha(\mathbf{x}), S\} = \sum_\beta \int d^d y\, \frac{\partial \psi_\alpha(\mathbf{x})}{\partial \phi_\beta(\mathbf{y})} \{\phi_\beta(\mathbf{y}), S\} = 0 \tag{4.34}$$

since

$$\{\phi_\beta(\mathbf{y}), S\} = 0 \tag{4.35}$$

Therefore

$$V_\alpha[\psi] = \{\psi_\alpha(\mathbf{x}), E\} \tag{4.36}$$

We find immediately by integrating the $\{\psi_\alpha(\mathbf{x}), \varepsilon(\mathbf{y})\}$ over $\mathbf{y}$ that

$$V_\rho(\mathbf{x}) = -\nabla_{\mathbf{x}} \cdot \mathbf{g}(\mathbf{x}) \tag{4.37}$$

$$V_{g_i}(\mathbf{x}) = -\sum_j \nabla^j_{\mathbf{x}} \sigma^R_{ij}(\mathbf{x}) \tag{4.38}$$

$$V_\varepsilon(\mathbf{x}) = -\nabla \cdot \mathbf{J}_\varepsilon(\mathbf{x}) \tag{4.39}$$

where the reversible part of the stress tensor is given by

$$\sigma^R_{ij} = P\delta_{ij} + \rho V_i V_j + \Delta_{ij} \tag{4.40}$$

and the reversible part of the energy current is given by

$$\mathbf{J}^R_\varepsilon = \mathbf{V}(\varepsilon + P) \tag{4.41}$$

This form of the streaming velocity must satisfy the "divergence" condition (2.23),

$$\sum_\alpha \int d^dx \frac{\delta}{\delta\psi_\alpha(\mathbf{x})} \{\psi_\alpha(\mathbf{x}), F\} = 0 \tag{4.42}$$

The first term in the sum is zero,

$$-\int d^dx \frac{\delta}{\delta\rho(\mathbf{x})} (\nabla_x \cdot \mathbf{g}(\mathbf{x})) = 0 \tag{4.43}$$

since $\mathbf{g}$ and ρ are independent variables. This leaves

$$\int d^dx \left(\sum_{ij} \frac{\delta}{\delta g_i(\mathbf{x})} \nabla^j_x \sigma^R_{ij}(x) + \frac{\delta}{\delta\varepsilon(\mathbf{x})} \nabla_x \cdot \mathbf{J}_\varepsilon(x) \right) = 0 \tag{4.44}$$

Since ε is a local function of $\mathbf{g}$, Δ_{ij} and P are independent of $\mathbf{g}$. This follows for Δ_{ij} directly from its definition (4.30) and (4.21). Using (4.21) in (4.27), one can rewrite the pressure in the form

$$P(\mathbf{x}) = \sum_\alpha \phi_\alpha(\mathbf{x}) \frac{\delta U}{\delta\phi_\alpha(\mathbf{x})} - u(\mathbf{x}) \tag{4.45}$$

where $U = \int d^dx\, u(\mathbf{x})$. It is clear from (4.45) that $P(\mathbf{x})$ is independent of $\mathbf{g}$ and the first term in (4.44) reduces to

$$\begin{aligned}
&\sum_{ij}\int d^d\mathbf{x}\,\frac{\delta}{\delta g_i(\mathbf{x})}\nabla_x^j\left[\frac{g_i(\mathbf{x})\,g_j(\mathbf{x})}{\rho(\mathbf{x})}\right]\\
&\quad=\sum_{ij}\int d^dx\,d^dy\,\frac{\delta}{\delta g_i(\mathbf{x})}\nabla_x^j\left[\frac{g_i(\mathbf{y})\,g_j(\mathbf{y})}{\rho(\mathbf{y})}\,\delta(\mathbf{x}-\mathbf{y})\right]\\
&\quad=\sum_{ij}\int d^dx\,d^dy\left[\frac{\delta(\mathbf{x}-\mathbf{y})\,g_j(\mathbf{y})}{\rho(\mathbf{y})}+\delta_{ij}\frac{\delta(\mathbf{x}-\mathbf{y})\,g_i(\mathbf{y})}{\rho(\mathbf{y})}\right]\nabla_{\mathbf{x}}^j\delta(\mathbf{x}-\mathbf{y})\\
&\quad=\sum_{j}\int d^dy\,\frac{g_j(\mathbf{y})}{\rho(\mathbf{y})}(d+1)\int d^dx\,\delta(\mathbf{x}-\mathbf{y})\,\nabla_{\mathbf{x}}^j\delta(\mathbf{x}-\mathbf{y})
\end{aligned}\tag{4.46}$$

The last integral in (4.46) is the same as the one that appears in (3.15), which vanishes,

$$\int d^dx\,\delta(\mathbf{x}-\mathbf{y})\,\nabla_{\mathbf{x}}^j\delta(\mathbf{x}-\mathbf{y})=0\tag{4.47}$$

The divergence condition (4.44) therefore reduces to

$$\int d^dx\,\frac{\delta}{\delta\varepsilon(\mathbf{x})}\nabla_{\mathbf{x}}\cdot\left\{\frac{\mathbf{g}(\mathbf{x})}{\rho(\mathbf{x})}\left[\varepsilon(\mathbf{x})+P(\mathbf{x})\right]\right\}=0\tag{4.48}$$

The first term is seen to be zero by following essentially the same analysis used to obtain (4.46). One is left then with

$$\int d^dx\,\frac{\delta}{\delta\varepsilon(\mathbf{x})}\nabla_x\cdot\left[\frac{\mathbf{g}(x)}{\rho(\mathbf{x})}P(\mathbf{x})\right]=0\tag{4.49}$$

This can be rewritten in the form

$$\int d^dx\,d^dy\,\frac{\mathbf{g}(\mathbf{y})}{\rho(\mathbf{y})}\frac{\delta P(\mathbf{y})}{\delta\varepsilon(\mathbf{x})}\nabla_x\delta(\mathbf{x}-\mathbf{y})=0\tag{4.50}$$

Since $P=P(\rho, s)$, we can write

$$\begin{aligned}
\frac{\delta P(\mathbf{y})}{\delta\varepsilon(\mathbf{x})}&=\int d^dz\sum_\alpha\frac{\delta P(\mathbf{y})}{\delta\phi_\alpha(\mathbf{z})}\frac{\delta\phi_\alpha(\mathbf{z})}{\delta\varepsilon(\mathbf{x})}\\
&=\int d^dz\,\frac{\delta P(\mathbf{y})}{\delta s(\mathbf{z})}\frac{\delta s(\mathbf{z})}{\delta\varepsilon(\mathbf{x})}
\end{aligned}\tag{4.51}$$

If ε and u local functions of s, then

$$\frac{\delta s(\mathbf{y})}{\delta\varepsilon(\mathbf{x})}=\frac{\partial s(\mathbf{x})}{\partial\varepsilon(\mathbf{x})}\delta(\mathbf{x}-\mathbf{z})\tag{4.52}$$

Since P is also a local function of s, via (4.27), we obtain

$$\frac{\delta P(\mathbf{y})}{\delta \varepsilon(\mathbf{x})} = \int d^d z \, \frac{\partial P(\mathbf{y})}{\partial s(\mathbf{y})} \, \delta(\mathbf{y}-\mathbf{z}) \, \frac{\delta s(\mathbf{x})}{\delta \varepsilon(\mathbf{x})} \, \delta(\mathbf{x}-\mathbf{z})$$
$$= \frac{\partial P(\mathbf{y})}{\partial s(\mathbf{y})} \frac{\partial s(\mathbf{y})}{\partial \varepsilon(\mathbf{y})} \, \delta(\mathbf{x}-\mathbf{y}) \tag{4.53}$$

Inserting (4.53) into (4.50), one finds, using (4.47), that the identity is satisfied. Thus we satisfy the divergence condition if E has the form (4.21) and u is a local function of s. Thus E can have nonlocal contributions only via its dependence on the mass density ρ.

These locality requirements reflect back on the results of Section 4.1. Reconsider the equations defining the intensive variables in the entropy representation (4.8). One can then use standard thermodynamic manipulations in treating derivatives with respect to $\mathbf{g}$ and ε to obtain

$$\tilde{G}_\varepsilon(\mathbf{x}) = \frac{\delta S}{\delta \varepsilon(\mathbf{x})} = \frac{1}{T(\mathbf{x})} \tag{4.54}$$

and

$$\tilde{G}_{g_i}(\mathbf{x}) = \frac{\delta S}{\delta g_i(\mathbf{x})} = -V_i \frac{1}{T} \tag{4.55}$$

Since ε may not be local in ρ, one cannot conclude that $\tilde{G}_\rho(\mathbf{x}) = -\mu(\mathbf{x})/T(\mathbf{x})$.

4.4. Dissipative Coefficients

The dissipative coefficients and the noise are specified in (2.1) by giving the matrix $G^{\alpha\sigma\gamma}[\mathbf{x}, \mathbf{y}; \psi]$ and the matrix $D_{\sigma\gamma,\mu\nu}$. In the case of a simple fluid where all of the variables $\psi_\alpha(\mathbf{x})$ are conserved, the most local form we can choose for G is given by

$$G^{\alpha\sigma\gamma}[\mathbf{x}, \mathbf{y}; \psi] = \sum_i \nabla_{\mathbf{x}}^i [f_i^{\alpha\sigma\gamma}(\psi(\mathbf{x})) \, \delta(\mathbf{x}-\mathbf{y})] \tag{4.56}$$

Inserting this result in (2.20) gives

$$\Gamma^{\alpha\beta}[\mathbf{x}, \mathbf{y}; \psi] = \sum_{i,j} \sum_{\sigma,\gamma} \sum_{\mu,\nu} \nabla_{\mathbf{x}}^i \nabla_{\mathbf{y}}^j [f_i^{\alpha\sigma\gamma}(\psi(\mathbf{x})) \, f_j^{\beta\mu\nu}(\psi(\mathbf{x})) \, D_{\sigma\gamma,\mu\nu} \delta(\mathbf{x}-\mathbf{y})]$$

The locality condition (2.19) then reduces to

$$\int d^d x \sum_{\alpha,i} \frac{\delta}{\delta \psi_\alpha(x)} \nabla_{\mathbf{x}}^i [f_i^{\alpha\sigma\gamma}(\psi) \, \delta(\mathbf{x}-\mathbf{y})] = \int d^d x \sum_{\alpha,i} \frac{\delta f_i^{\alpha\sigma\gamma}(\mathbf{y})}{\delta \psi_\alpha(\mathbf{x})} \nabla_{\mathbf{x}}^i \delta(\mathbf{x}-\mathbf{y}) = 0 \tag{4.57}$$

If $f^{\alpha\sigma\gamma}(\mathbf{x})$ is a local function of the $\psi_\alpha(\mathbf{x})$, then

$$\frac{\delta f_i^{\alpha\sigma\gamma}(\mathbf{y})}{\delta\psi_\alpha(\mathbf{x})}=\frac{\partial f^{\alpha\sigma\gamma}(\mathbf{y})}{\partial\psi_\alpha(\mathbf{y})}\,\delta(\mathbf{x}-\mathbf{y}) \tag{4.58}$$

and (4.57) is satisfied through the use of (4.58) and (4.47).

A key point here is that there is no dissipative coefficient associated with the mass density, so

$$f_i^{\rho\sigma\gamma}=0 \tag{4.59}$$

and the locality conditions therefore require only that $f_i^{\alpha\sigma\gamma}$ be a local function of **g** and ε. As will be demonstrated below, the choice which leads to the standard constituitive relations is

$$f_i^{\alpha\sigma\gamma}(\mathbf{x})=\delta_{\gamma,g_i}[T(\mathbf{x})]^{1/2}\left\{\delta_{\sigma,\alpha}[\delta_{\alpha,g_j}+[T(\mathbf{x})]^{1/2}\,\delta_{\alpha,\varepsilon}]+\delta_{\alpha,\varepsilon}\sum_k\delta_{\sigma,g_k}V_k(\mathbf{x})\right\} \tag{4.60}$$

Since the entropy is assumed to be a local function of ε and **g**, so also are the derivatives of the entropy with respect to ε and **g**, T^{-1} and $-T^{-1}V$, respectively, local functions of **g** and ε. Thus, the choice that S be a local function of **g** and ε allows one to satisfy both the divergence and locality conditions.

Finally the dissipative matrix $D_{\sigma\gamma,\mu\nu}$ are

$$D_{g_i g_j,\, g_k g_l}=\frac{\eta_{ij,kl}}{T_0} \tag{4.61}$$

and

$$D_{g_i\varepsilon,\, g_j\varepsilon}=\frac{\lambda_0}{T_0}\,\delta_{ij} \tag{4.62}$$

where $\eta_{ij,kl}$ is the usual viscosity matrix for an isotropic system

$$\eta_{ij,kl}=\eta(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\tfrac{1}{3}\delta_{ij}\delta_{kl})+\zeta\delta_{ij}\delta_{kl} \tag{4.63}$$

with η and ζ the shear and bulk viscosities, and λ_0 in (4.62) is the thermal conductivity.

Using (4.60)–(4.62) in (2.20), one easily finds the kinetic coefficients:

$$\Gamma^{g_i g_j}[\mathbf{x},\mathbf{y};\psi]=\sum_{k,l}\nabla_x^k\nabla_y^l\left[\eta_{ik,jl}\frac{T(x)}{T_0}\,\delta(\mathbf{x}-\mathbf{y})\right] \tag{4.64}$$

$$\Gamma^{g_i\varepsilon}[\mathbf{x},\mathbf{y};\psi]=\sum_{j,k,l}\nabla_x^k\nabla_y^l\left[\eta_{ik,jl}V_j(\mathbf{x})\frac{T(\mathbf{x})}{T_0}\delta(\mathbf{x}-\mathbf{y})\right] \tag{4.65}$$

$$\begin{aligned}\Gamma^{\varepsilon\varepsilon}[\mathbf{x},\mathbf{y};\psi]&=\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}\left[\lambda_0\frac{T^2(\mathbf{x})}{T_0}\delta(\mathbf{x}-\mathbf{y})\right]\\&\quad+\sum_{ijkl}\nabla_{\mathbf{x}}^k\nabla_{\mathbf{y}}^l\left[\eta_{ik,jl}V_i(\mathbf{x})\,V_j(\mathbf{x})\frac{T(x)}{T_0}\delta(\mathbf{x}-\mathbf{y})\right]\end{aligned} \tag{4.66}$$

It is useful to see explicitly how these results reproduce the expected constitutive relations. One has first from the momentum equation

$$\begin{aligned}\sum_k\nabla_x^k\sigma_{ik}(\mathbf{x})&=\sum_\beta\int d^dy\,\Gamma^{g_i\beta}[\mathbf{x},\mathbf{y};\psi]\frac{\delta F}{\delta\psi_\beta(\mathbf{y})}\\&=\int d^dy\left(\sum_j\Gamma^{g_ig_j}[\mathbf{x},\mathbf{y};\psi]\frac{\delta F}{\delta g_j(\mathbf{y})}+\Gamma^{g_i\varepsilon}[\mathbf{x},\mathbf{y};\psi]\frac{\delta F}{\delta\varepsilon(\mathbf{y})}\right)\end{aligned} \tag{4.67}$$

Using the thermodynamic relations

$$\frac{\delta F}{\delta g_j(\mathbf{y})}=\frac{T_0}{T(\mathbf{y})}V_j(\mathbf{y}) \tag{4.68}$$

$$\frac{\delta F}{\delta\varepsilon(\mathbf{y})}=1-\frac{T_0}{T(\mathbf{y})} \tag{4.69}$$

(4.67) becomes

$$\begin{aligned}&\sum_k\nabla_{\mathbf{x}}^k\sigma_{ik}(\mathbf{x})\\&\quad=\sum_{jkl}\int d^dy\,\nabla_x^k\nabla_y^l\left[\eta_{ik,jl}\frac{T(\mathbf{x})}{T_0}\delta(\mathbf{x}-\mathbf{y})\right]\frac{T_0}{T(\mathbf{y})}V_j(\mathbf{y})\\&\qquad+\sum_{jkl}\int d^dy\,\nabla_x^k\nabla_y^l\left[\eta_{ik,jl}V_0(\mathbf{x})\frac{T(\mathbf{x})}{T_0}\delta(\mathbf{x}-\mathbf{y})\right]\left(1-\frac{T_0}{T(\mathbf{y})}\right)\\&\quad=-\sum_{jkl}\nabla_x^k\eta_{ik,jl}\left[T(\mathbf{x})\nabla_x^l\frac{V_j(\mathbf{x})}{T(\mathbf{x})}-V_j(\mathbf{x})\,T(\mathbf{x})\nabla_x^l\left(\frac{1}{T(x)}\right)\right]\end{aligned} \tag{4.70}$$

and one can identify

$$\sigma_{ik}(\mathbf{x})=-\sum_{il}\eta_{ik,jl}\nabla_x^lV_j(\mathbf{x}) \tag{4.71}$$

as expected. The energy equation gives

$$
\begin{aligned}
\boldsymbol{\nabla}_x \cdot \mathbf{J}_\varepsilon(\mathbf{x}) &= \sum_\beta \int d^d y\, \Gamma^{\varepsilon\beta}[\mathbf{x}, \mathbf{y}; \psi]\, \frac{\delta F}{\delta \psi_\beta(\mathbf{y})} \\
&= \sum_{ikjl} \int d^d y\, \nabla_x^k \nabla_y^l \left[\eta_{ik,jl} V_i(\mathbf{x}) \frac{T(\mathbf{x})}{T_0} \delta(\mathbf{x}-\mathbf{y}) \right] \frac{T_0}{T(\mathbf{y})} V_j(\mathbf{y}) \\
&\quad + \sum_{ikjl} \int d^d y\, \nabla_x^k \nabla_y^l \left[\eta_{ik,jl} V_i(\mathbf{x})\, V_j(\mathbf{x}) \frac{T(\mathbf{x})}{T_0} \delta(\mathbf{x}-\mathbf{y}) \right] \left(1 - \frac{T_0}{T(\mathbf{y})} \right) \\
&\quad + \int d^d y\, \boldsymbol{\nabla}_x \cdot \boldsymbol{\nabla}_y \left[\lambda_0 \frac{T^2(\mathbf{x})}{T_0} \delta(\mathbf{x}-\mathbf{y}) \right] \left(1 - \frac{T_0}{T(\mathbf{y})} \right) \\
&= -\sum_{ikjl} \nabla_x^k \left\{ \eta_{ik,jl} V_i(\mathbf{x})\, T(\mathbf{x}) \left[\nabla_x^l \frac{V_j(\mathbf{x})}{T(\mathbf{x})} - V_j(\mathbf{x}) \nabla_{\mathbf{x}}^l \left(\frac{1}{T(\mathbf{x})} \right) \right] \right\} \\
&\quad + \boldsymbol{\nabla}_x \cdot \left[\lambda_0 T^2(\mathbf{x}) \boldsymbol{\nabla}_{\mathbf{x}} \left(\frac{1}{T(\mathbf{x})} \right) \right]
\end{aligned}
$$

and one has immediately that

$$
J_\varepsilon^i(\mathbf{x}) = -\sum_{kjl} \eta_{ik,jl} V_i(\mathbf{x}) \nabla_x^l V_j(\mathbf{x}) - \lambda_0 \nabla_x^i T(\mathbf{x}) \tag{4.72}
$$

and one recovers Fourier's law when $\mathbf{V} = 0$.

4.5. Summary for a Normal Fluid

Putting all of these results together, one obtains the equations of nonlinear fluctuating hydrodynamics for a normal fluid:

$$
\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{g} \tag{4.73}
$$

$$
\begin{aligned}
\frac{\partial g_i}{\partial t} &= -\nabla^i P - \sum_j \nabla^j \left[\left(\frac{g_i g_j}{\rho} \right) + \Delta_{ij} \right] + \sum_{jkl} \eta_{ij,kl} \nabla^j \nabla^k V_l \\
&\quad + \sum_j \nabla^j (\sqrt{T}\, \theta_{ij})
\end{aligned} \tag{4.74}
$$

$$
\begin{aligned}
\frac{\partial \varepsilon}{\partial t} &= -\sum_i \nabla^i [V_i(\varepsilon + P)] + \lambda_0 \nabla^2 T + \sum_{ijkl} \eta_{ij,kl} \nabla_i (V_j \nabla^k V_l) \\
&\quad + \sum_{i,j} \nabla^i (\sqrt{T}\, V_j \theta_{ij}) + \sum_i \nabla^i (T \zeta_i)
\end{aligned} \tag{4.75}
$$

where the Gaussian noise terms satisfy second moments

$$\langle \theta_{ij}(\mathbf{x}, t)\, \theta_{kl}(\mathbf{y}, t') \rangle = 2\eta_{ij,kl}\delta(\mathbf{x}-\mathbf{y})\, \delta(t-t') \tag{4.76}$$

and

$$\langle \zeta_i(\mathbf{x}, t)\, \zeta_j(\mathbf{y}, t') \rangle = 2\lambda_0 \delta_{ij}\delta(\mathbf{x}-\mathbf{y})\, \delta(t-t') \tag{4.77}$$

The various ingredients in (4.73)–(4.75) can be obtained from the energy functional assumed to be of the form (4.21) and local in ε. The pressure is given by (4.45). The quantity Δ_{ij} is defined by (4.30) and indicates that we have assumed that E is nonlocal with respect to the mass density and therefore takes the form

$$\Delta_{ij} = \frac{\partial \varepsilon(x)}{\delta(\nabla^j \rho(x))} \nabla^i \rho(x) \tag{4.78}$$

where $\varepsilon = \varepsilon(s, \mathbf{g}, \rho, \nabla\rho)$. Finally, the local velocity and temperature are given by (4.5) and (4.4).

Neglecting the noise terms and the term Δ_{ij}, one regains the equations of Landau and Lifshitz.$^{(8)}$ Neglecting all gradient terms, one returns to the equations first obtained by Morozov.$^{(13)}$

4.6. The Construction of the Energy Functional

In this section, we give an example of the type of free energy functional which is of the form (4.21) and which may be useful in the study of glassy dynamics and solidification.

We begin with the well-known result for the ideal gas contribution to the entropy given by

$$S_{\mathrm{id}} = \int d^d x \left\{ -\left(\frac{d}{2}+1\right) \frac{\rho(\mathbf{x})}{m} \left[\ln\left(\frac{\rho(\mathbf{x})}{\rho_0}\right) - 1 \right] + \frac{d}{2} \frac{\rho(\mathbf{x})}{m} \ln\left(\frac{u(\mathbf{x})}{u_0}\right) \right\} \tag{4.79}$$

where u_0 is the ambient internal energy density. We can then obtain from (4.79) the ideal gas part of the internal energy density $u_{\mathrm{ideal}}(x)$ in terms of ρ and s of the form

$$u_{\mathrm{ideal}}(x) = u_0 \exp([2ms(x)/d\rho(\mathbf{x})] + (1+2/d)\{\ln[\rho(\mathbf{x})/\rho_0] - 1\})$$

If we then add to $u_{\mathrm{ideal}}(x)$ an "interaction" contribution $f(\rho, \nabla\rho)$ to the internal energy density, we obtain the free energy functional,

$$\begin{aligned} E = \int d^d x \frac{\mathbf{g}^2(x)}{2\rho(x)} + \int d^d x \Bigg(u_0 \exp \Bigg\{ \frac{s(\mathbf{x})}{\frac{1}{2} d[\rho(\mathbf{x})/m]} \\ + \left(1+\frac{2}{d}\right)\left[\ln\left(\frac{\rho(\mathbf{x})}{\rho_0}\right) - 1 \right] \Bigg\} \Bigg) + \int d^d x\, f(\rho, \nabla\rho) \end{aligned} \tag{4.80}$$

It is then straightforward to show that the intensive variables are given by

$$T = 2mu_{\text{ideal}}/d\rho = 2m(\varepsilon - g^2/2\rho - f)/d\rho \tag{4.81}$$

and

$$P = \frac{\rho T}{m} + \left(\rho \frac{\partial f}{\partial \rho} - f\right) + h(\nabla\rho)^2 - \nabla \cdot \left[\frac{h}{2} \nabla\rho^2\right] \tag{4.82}$$

where h is defined by

$$\frac{\partial f}{\partial(\nabla^i \rho)} = (\nabla^i \rho)\, h(\rho, \nabla\rho) \tag{4.83}$$

These results then completely specify the hydrodynamic equations in terms of ρ, $\mathbf{g}$, and ε.

5. CONCLUSIONS

We have shown that the development due to Zubarev and Morozov(12) and Morozov(13) can be incorporated into a set of conventional nonlinear Langevin equations with multiplicative noise. In the process, we also have shown that in order for the multiplicative noise to reconcile both thermodynamics and phenomenological laws for the dissipative fluxes, the driving free energy should be a local function of the momentum and energy variables (no gradient terms) and can contain nonlocal and/or gradient terms of the mass density only.

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